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STRENGTH OF PRESTRESSED CONCRETE PAVEMENTS

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SYNOPSIS

A theoretical method is developed for determining the distribution of stresses and deflections in prestressed concrete pavements beyond conditions to which the elastic theory is applicable. The method is limited to a load applied at an interior position of a slab supported by a dense liquid foundation and prestressed equally in the longitudinal and transverse directions. By use of this method, the magnitude of load causing top surface cracking and the location of this cracking can be predicted.

INTRODUCTION

In studying the performance of prestressed concrete pavements, it is necessary to investigate stresses and deflections beyond conditions to which the elastic theory is applicable. Theoretical studies beyond the elastic range have been reported for the case of interior loading by Levi[2] and by Cot and Beck-

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2 "Étude Théorique Experimentale d'une Dalle Précontrainte sur Appui Elastique au-dela des Limites d'Elasticité," by M. Franco Lévi, Annales de l'Institut Technique du Batiment at des Travaux Publics, No. 66, June, 1953.

er.[3] Their methods differ principally in the type of cracking assumed at the bottom surface of the slab. Lévi assumed an initial circular crack, and Cot and Becker assumed a number of radial cracks. The radial crack assumption is more reasonable, because the elastic theory shows the tangential moment to be larger than the radial moment near the loading point, as shown by Fig. 1. Furthermore, radial cracking was observed in load tests of prestressed concrete slabs at the Portland Cement Association (PCA) Laboratories and at the Corps of Engineers Ohio River Division Laboratories.[4] Based on the radial crack assumption, a step-by-step method is developed herein whereby an increment of load results in an incremental increase in the crack length.

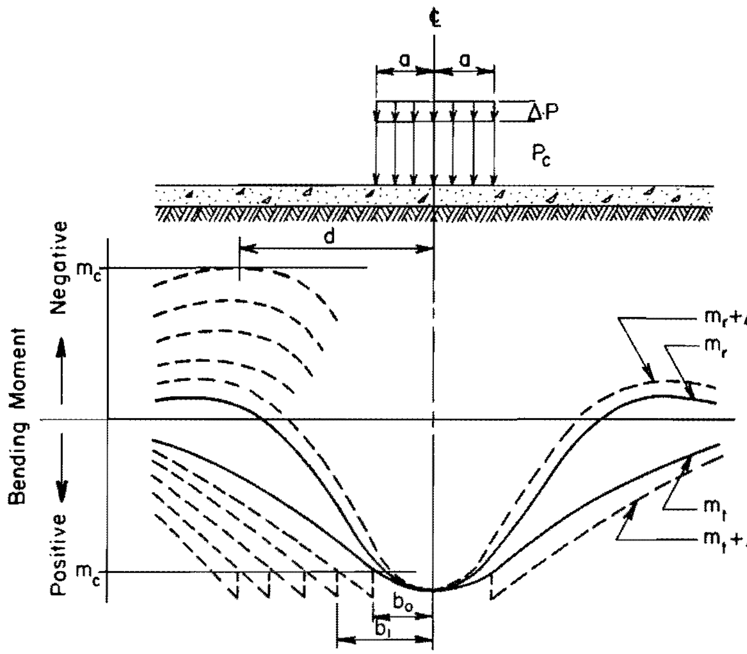

FIG. 1.—ELASTIC AND PLASTIC MOMENT DISTRIBUTION

In contrast, the Cot and Becker solution selects a length for the bottom radial crack and determines the load required to produce a crack of this length.

The method is not proposed as a design procedure, but rather as a contribution toward a better understanding of the performance of prestressed pavements.

[3] "Calcul des Pistes en Beton Précontraint," by P. D. Cot and E. Becker, Revue Générale des Routes et des Aerodromes, Paris, France, No. 292, May, 1956.

[4] "Model Studies of Prestressed Rigid Pavements for Airfields," by P. F. Carlton and R. M. Behrmann, Highway Research Bd. Bulletin 179, Washington, D. C., 1958.

Notation.—The letter symbols adopted for use in this paper are defined where they first appear and are arranged alphabetically in Appendix II.

GENERAL CONSIDERATIONS

The analysis is limited to a slab supported by a "dense liquid" subgrade and prestressed equally in the longitudinal and transverse directions. The "dense liquid" concept assumes that the deflection of the subgrade at any point is proportional to the vertical load per unit area imposed at that point, but independent of the vertical load at any other point. The slab is subjected to a load distributed uniformly over a circular area in the interior at a considerable distance from an edge. For magnitudes of load not sufficient to cause bottom surface cracking, the slab behaves elastically, and the solutions for the moments and deflections have already been given by Westergaard and others.[5,6] The equations in the form needed for the present solution are shown in Appendix I.

The bottom cracks begin to form under the load when the positive moment equals the cracking moment m_c. A small increase in load ΔP beyond that initiating bottom cracking will cause these cracks to extend radially from the center of the load. A circular area is specified defining the limits of these cracks, as shown in Fig. 2. Assuming no resistance to moment in the tangential direction within this area, the moment distribution for the load increment ΔP is shown in Fig. 3. By adding these moments to the moment distributions for the load P_c initiating the bottom cracking, the total moment distributions beyond the elastic range are obtained, as shown in Fig. 1. The addition of negative radial moments and positive tangential moments indicates that the radial cracks will become longer with additional load and that circular cracking will not occur in the bottom. Next, the longer radial cracks are assumed, and the moment distributions are determined for the second load increment. More negative radial moments and positive tangential moments are added. As the load is increased, the bottom radial cracks are extended, and the point of zero radial moment moves nearer the center of loading. Beyond this point, the negative radial moment increases more rapidly in magnitude until finally after several load increments it will become equal to the cracking moment, which results in a circular crack in the top surface of the slab. The load causing this top cracking is assumed for this analysis as the maximum that can be allowed on a prestressed concrete pavement. The theoretical analysis is developed to determine this load and to indicate the location of the top surface crack.

ASSUMPTIONS

The following assumptions are made: (1) The intensity of the subgrade reaction at each point of the bottom surface of the slab is proportional to the

[5] "Stresses in Concrete Pavements Computed by Theoretical Analysis," by H. M. Westergaard, Public Roads, April, 1926, p. 25.

[6] "Deflections, Moments, and Reactive Pressures for Concrete Pavements," by G. Pickett, M. E. Raville, W. C. Janes, and F. J. McCormick., Kansas State College Bulletin No. 65, Manhattan, Kansas, 1951.

deflection of the slab at that point; (2) the moment-curvature graph is represented by two straight lines, as indicated in Fig. 4; and (3) the radial cracks in the bottom surface of the concrete slab extend from the center of the load within a circular area which becomes larger as the load is increased.

The circular area within which the radial cracks form is defined as the "cracked zone." After cracking occurs, the slab will be inelastic within the cracked zone and elastic outside the cracked zone. Separate solutions are provided for the cracked zone and the uncracked zone. Final expressions for moment, shear, and deflection are obtained by considering the necessary continuity at the common boundary between the two zones.

CRACKED ZONE

Based on the assumption (illustrated in Fig. 4) that the moment remains constant for increased curvature beyond the cracking moment, the slab has no rigidity in the tangential direction within the cracked zone. The radial moments and shears are assumed to be uniformly distributed along the periphery of each circle concentric with the center of the load. The problem is thereby reduced to the analysis of a wedge-shaped beam that is free at the apex and connected to the uncracked slab at the opposite end. Because a rigorous solution is rather difficult, an approximation is introduced by assuming that the subgrade reaction in the cracked zone decreases proportionately with the distance from the center of the load, as shown in Fig. 5. This assumption agrees with test data obtained at the PCA Laboratories and does not induce any serious error in determining the moment distributions if the cracked area is small.

The deflection y at any point in the cracked zone can be approximated by

$$y = y_b + \Theta_b(b - x) \qquad \text{. . . . . (1)}$$

in which x is the distance from the apex, b represents the radius of the cracked zone, Θ_b is the slope of the deflection line, and y_b denotes the deflection at x = b (see Fig. 5).

The increment of load per unit area Δp is applied to the wedge-shaped beam on the sector of the circular area having a radius equal to a, measured from the apex. Letting the angle at the apex equal one radian, the radial bending moment M_o inside the loading area ($0 \leq x \leq a$) caused by the applied load is

$$M_o = - \int_0^x \Delta p \, (x - r) \, r \, dr = - \frac{\Delta p \, x^3}{6} \qquad \text{. . . . . (2)}$$

The radial bending moment outside the loading area ($a \leq x \leq b$) caused by the applied load is

$$M_o = \frac{-\Delta p \, a^2}{2} \left(x - \frac{2a}{3} \right) \qquad \text{. . . . . . . (3)}$$

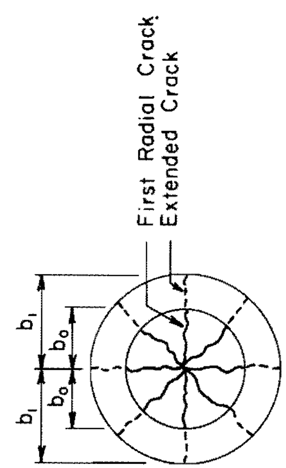

FIG. 2.—ASSUMED BOTTOM SURFACE CRACK PATTERN

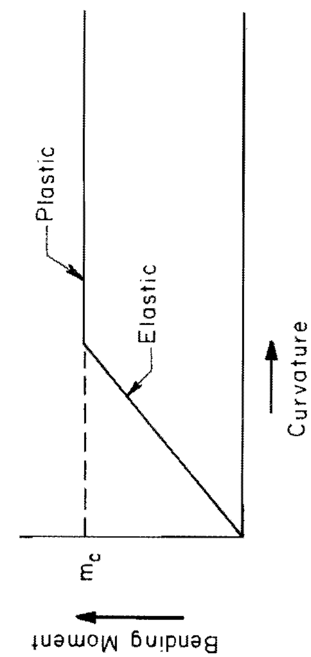

FIG. 4.—ASSUMED MOMENT AND CURVATURE RELATIONSHIP

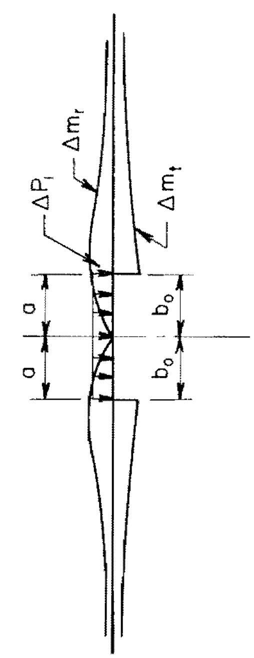

FIG. 3.—MOMENT DISTRIBUTION DUE TO LOAD INCREMENT BEYOND ELASTIC RANGE

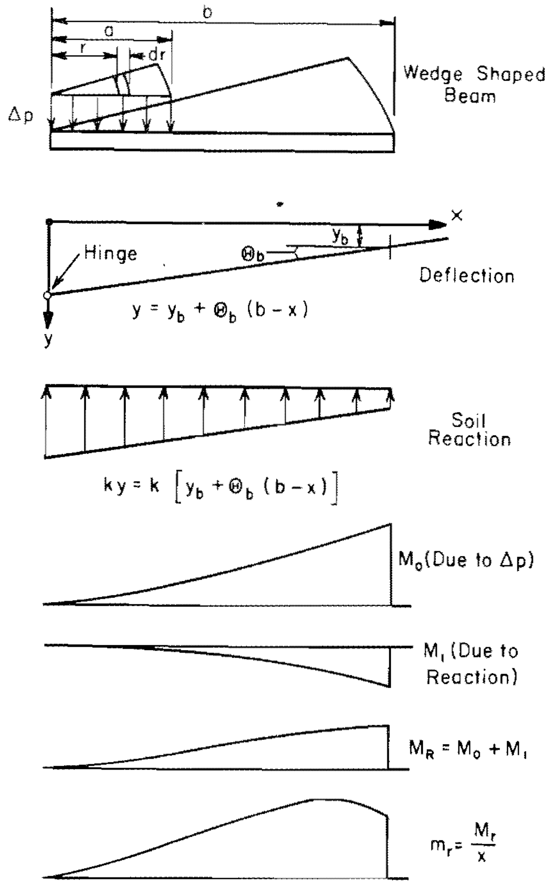

FIG. 5.—DEFLECTION, SOIL REACTION, AND MOMENT DISTRIBUTION INSIDE CRACKED ZONE

The radial bending moment M_1 due to subgrade reaction is

$$M_1 = \int_0^x k\, y\, r\, (x - r)\, dr = \int_0^x k \left[y_b + \Theta_b (b - r)\right] r (x - r)\, dr$$

$$= \frac{k x^3}{6} y_b + \frac{k x^3}{6} \left(b - \frac{x}{2}\right) \Theta_b \quad \ldots\ldots\ldots \quad (4)$$

in which k represents the modulus of subgrade reaction.

Therefore, the total radial bending moment M_r is

$$M_r = M_o + M_1$$

$$= \begin{bmatrix} -\dfrac{\Delta p\, x^3}{6} \\ \\ -\dfrac{\Delta p\, a^2}{2}\left(x - \dfrac{2}{3} a\right) \end{bmatrix} + \frac{k x^3}{6} y_b + \frac{k x^3}{6}\left(b - \frac{x}{2}\right)\Theta_b \text{ for } \begin{bmatrix} 0 \le x \le a \\ \\ a \le x \le b \end{bmatrix} \quad \ldots \quad (5)$$

The radial bending moment per unit length of arc m_r is

$$m_r = \frac{M_r}{x} =$$

$$\begin{bmatrix} -\dfrac{\Delta p\, x^2}{6} \\ \\ -\dfrac{\Delta p\, a^2}{2}\left(1 - \dfrac{2}{3}\dfrac{a}{x}\right) \end{bmatrix} + \frac{k x^2}{6} y_b + \frac{k x^2}{6}\left(b - \frac{x}{2}\right)\Theta_b \text{ for } \begin{bmatrix} 0 \le x \le a \\ \\ a \le x \le b \end{bmatrix} \quad \ldots \quad (6)$$

The relationship between the total load increment ΔP for the entire circular area and the distributed load increment Δp is

$$\Delta P = \pi a^2 \Delta p \quad \ldots\ldots\ldots\ldots\ldots \quad (7)$$

The following nondimensional quantities are introduced:

$$\xi = \frac{x}{L} \quad \ldots\ldots\ldots\ldots\ldots \quad (8a)$$

$$a = \frac{a}{L} \quad \ldots\ldots\ldots\ldots\ldots \quad (8b)$$

and

$$\beta = \frac{b}{L} \quad \ldots\ldots\ldots\ldots\ldots \quad (8c)$$

in which

$$L = \sqrt[4]{\frac{E h^3}{12\left(1 - \mu^2\right) k}} \quad \ldots\ldots\ldots\ldots \quad (8d)$$

and h is the thickness of the slab, E represents the elastic modulus of the concrete, and μ denotes the Poisson's ratio of the concrete.

By using Eqs. 7 and 8, the moment is

$$m_r = \begin{bmatrix} -\dfrac{\Delta P\,\xi^2}{6\,\pi\,\alpha^2} \\ \\ -\dfrac{\Delta P}{2\,\pi}\left(1-\dfrac{2}{3}\,\dfrac{\alpha}{\xi}\right) \end{bmatrix} + \frac{k\,L^2\xi^2}{6}\,y_b + \frac{k\,L^3\,\beta\xi^2}{6}\left(1-\frac{\xi}{2\,\beta}\right)\Theta_b \text{ for } \begin{bmatrix} 0\leq\xi\leq\alpha \\ \\ \alpha\leq\xi\leq\beta \end{bmatrix} \quad \ldots (9)$$

At the apex where $x = \xi = 0$, the moment is

$$m_r = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

At the opposite end where $x = b$ or $\xi = \beta$, the moment is

$$m_r = -\frac{\Delta P}{2\pi}\left(1-\frac{2}{3}\,\frac{\alpha}{\beta}\right) + k\,L^2\left(\frac{\beta^2}{6}\,y_b + \frac{\beta^3}{12}\,\Theta_b\,L\right) \ldots \quad (11)$$

The shearing force per unit length of arc v, obtained in a similar manner, is

$$v = \begin{bmatrix} -\dfrac{\Delta P\xi}{2\,\pi\,L\,\alpha^2} \\ \\ -\dfrac{\Delta P}{2\,\pi\,L\xi} \end{bmatrix} + \frac{k\,L\,\xi}{2}\,y_b + \frac{k\,L^2\beta\,\xi}{2}\left(1-\frac{2}{3}\,\frac{\xi}{\beta}\right)\Theta_b \text{ for } \begin{bmatrix} 0\leq\xi\leq\alpha \\ \\ \alpha\leq\xi\leq\beta \end{bmatrix} \ldots (12)$$

At the apex where $x = \xi = 0$, the shear is

$$v = 0$$

At the opposite end where $x = b$ or $\xi = \beta$, the shear is

$$v = -\frac{\Delta P}{2\,\pi\,L\beta} + k\,L\left(\frac{\beta}{2}\,y_b + \frac{\beta^2}{6}\,\Theta_b\,L\right) \quad \ldots\ldots \quad (14)$$

UNCRACKED ZONE

Outside the cracked zone, the slab can be analyzed as an elastically supported infinite flat plate having a circular hole around which uniformly distributed shear and moment are applied. The solutions for moment and shear

must satisfy the conditions of continuity at the edge of the hole or boundary of the cracked zone.

The deflection of the slab must satisfy the following differential equation.[7,8]

$$D\left(\frac{d^2}{dx^2} + \frac{1}{x}\,\frac{d}{dx}\right)\left(\frac{d^2y}{dx^2} + \frac{1}{x}\,\frac{dy}{dx}\right) + ky = 0 \quad \ldots \quad (15)$$

Since

$$D = k\,L^4 \quad \ldots\ldots\ldots\ldots \quad (16a)$$

and

$$\frac{x}{L} = \xi \quad \ldots\ldots\ldots\ldots \quad (16b)$$

$$\left(\frac{d^2}{d\xi^2} + \frac{1}{\xi}\,\frac{d}{d\xi}\right)\left(\frac{d^2y}{d\xi^2} + \frac{1}{\xi}\,\frac{dy}{d\xi}\right) + y = 0 \quad \ldots \quad (16c)$$

The general solution of Eq. 16c is given by Hetényi in terms of the Z-functions[8] as follows:

$$y = C_1\,Z_1\,(\xi) + C_2\,Z_2\,(\xi) + C_3\,Z_3\,(\xi) + C_4\,Z_4\,(\xi) \ldots \quad (17)$$

in which $Z_1\,(\xi)$ represents the real part of $J_o\,(\xi\sqrt{i})$, $Z_2\,(\xi)$ denotes the imaginary part of $J_o\,(\xi\sqrt{i})$, $Z_3\,(\xi)$ represents the real part of $H_o^{(1)}\,(\xi\sqrt{i})$, and $Z_4\,(\xi)$ denotes the imaginary part of $H_o^{(1)}\,(\xi\sqrt{i})$. Since the slab is assumed to be of infinite extent, the deflection, y, and slope, $\frac{dy}{dx}$, approach zero as x approaches infinity. This yields

$$C_1 = C_2 = 0 \quad \ldots\ldots\ldots\ldots \quad (18)$$

Then, the solution yields

$$y = C_3\,Z_3\,(\xi) + C_4\,Z_4\,(\xi) = A\,Z_3\,(\xi) + B\,Z_4\,(\xi) \quad \ldots \quad (19)$$

Accordingly,

$$L\,\frac{dy}{dx} = A\,Z_3'\,(\xi) + B\,Z_4'\,(\xi) \quad \ldots\ldots \quad (20)$$

7 "Theory of Plates and Shells," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1940.

8 "Beams on Elastic Foundation," by M. Hetényi, Univ. of Michigan Press, Ann Arbor, Mich., 1946.

and

$$L^2 \frac{d^2y}{dx^2} = A\left\{Z_4(\xi) - \frac{1}{\xi} Z_3'(\xi)\right\} + B\left\{-Z_3(\xi) - \frac{1}{\xi} Z_4'(\xi)\right\} \quad . . \quad (21)$$

The radial moment per unit length is

$$m_r = -D\left(\frac{d^2y}{dx^2} + \frac{\mu}{x}\frac{dy}{dx}\right)$$

$$= \frac{D}{L^2}\left[A\left\{-Z_4(\xi) + \frac{1-\mu}{\xi} Z_3'(\xi)\right\} + B\left\{Z_3(\xi) + \frac{1-\mu}{\xi} Z_4'(\xi)\right\}\right] \quad . . . \quad (22)$$

The tangential moment per unit length is

$$m_t = -\frac{D}{L^2}\left[A\left\{\mu Z_4(\xi) + \frac{1-\mu}{\xi} Z_3'(\xi)\right\} + B\left\{-\mu Z_3(\xi) + \frac{1-\mu}{\xi} Z_4'(\xi)\right\}\right] \quad . . . \quad (23)$$

The shearing force per unit length is

$$v = \frac{D}{L^3}\left[-A\, Z_4'(\xi) + B\, Z_3'(\xi)\right] \quad . . . . \quad (24)$$

At the boundary $(\xi = \beta)$

$$y = A\, Z_3(\beta) + B\, Z_4(\beta) \quad . . . . \quad (25)$$

$$\frac{dy}{dx} = \frac{A}{L} Z_3'(\beta) + \frac{B}{L} Z_4'(\beta) \quad . . . . \quad (26)$$

$$m_r = \frac{D}{L^2}\left[A\left\{-Z_4(\beta) + \frac{1-\mu}{\beta} Z_3'(\beta)\right\} + B\left\{Z_3(\beta) + \frac{1-\mu}{\beta} Z_4'(\beta)\right\}\right]. \quad (27)$$

and

$$v = \frac{D}{L^3}\left[-A\, Z_4'(\beta) + B\, Z_3'(\beta)\right] \quad . . . . \quad (28)$$

BOUNDARY OF THE ZONES

At the boundary, the deflections, slopes, radial moments, and shears for inside and outside the cracked zone must be the same. From Eqs. 11, 14, 25, 26, 27, and 28,

$$y_b = A\, Z_3(\beta) + B\, Z_4(\beta) \quad . . . . \quad (29)$$

$$\Theta_b L = -\left[A\, Z_3'(\beta) + B\, Z_4'(\beta)\right] \quad . . . . \quad (30)$$

$$-\frac{\Delta P}{2\pi}\left(1 - \frac{2}{3}\frac{\alpha}{\beta}\right) + \frac{D}{L^2}\left(\frac{\beta^2}{6} y_b + \frac{\beta^3}{12}\Theta_b L\right) = \frac{D}{L^2}\left[A\left\{-Z_4(\beta) + \frac{1-\mu}{\beta} Z_3'(\beta)\right\} + B\left\{Z_3(\beta) + \frac{1-\mu}{\beta} Z_4'(\beta)\right\}\right] \quad . . . . \quad (31)$$

and

$$-\frac{\Delta P}{2\pi\, L\beta} + \frac{D}{L^3}\left(\frac{\beta}{2} y_b + \frac{\beta^2}{6}\Theta_b L\right) = \frac{D}{L^3}\left[-A\, Z_4'(\beta) + B\, Z_3'(\beta)\right] \quad . . \quad (32)$$

By solving Eqs. 29 through 32 simultaneously, expressions for A, B, y_b, and Θ_b can be obtained.

FINAL EXPRESSIONS

Inside the cracked zone:

$$y = y_b + \Theta_b L\, (\beta - \xi) \quad . . . . \quad (33)$$

$$\frac{dy}{dx} = -\Theta_b \quad . . . . \quad (34)$$

$$m_r = \begin{bmatrix} -\dfrac{\Delta P\, \xi^2}{6\pi\, \alpha^2} \\ -\dfrac{\Delta P}{2\pi}\left(1 - \dfrac{2}{3}\dfrac{\alpha}{\xi}\right) \end{bmatrix} + \frac{k\, L^2 \xi^2}{6} y_b + \frac{k\, L^3 \beta \xi^2}{6}\left(1 - \frac{\xi}{2\beta}\right)\Theta_b \text{ for } \begin{bmatrix} 0 \le \xi \le \alpha \\ \alpha \le \xi \le \beta \end{bmatrix} \quad . . \quad (35)$$

$$m_t = 0 \quad . . . . \quad (36)$$

and

$$v = \begin{bmatrix} -\dfrac{\Delta P \xi}{2\pi L \alpha^2} \\ \\ -\dfrac{\Delta P}{2\pi L \xi} \end{bmatrix} + \frac{k L \xi}{2} y_b + \frac{k L^2 \beta \xi}{2}\left(1 - \frac{2}{3}\frac{\xi}{\beta}\right)\Theta_b \text{ for } \begin{bmatrix} 0 \leq \xi \leq \alpha \\ \\ \alpha \leq \xi \leq \beta \end{bmatrix} \quad \ldots \quad (37)$$

Outside the cracked zone:

$$y = A\, Z_3(\xi) + B\, Z_4(\xi) \quad \ldots\ldots\ldots\ldots \quad (38)$$

$$\frac{dy}{dx} = \frac{A}{L} Z_3'(\xi) + \frac{B}{L} Z_4'(\xi) \quad \ldots\ldots\ldots\ldots \quad (39)$$

$$m_r = \frac{D}{L^2}\left[A\left\{-Z_4(\xi) + \frac{1-\mu}{\xi} Z_3'(\xi)\right\} + B\left\{Z_3(\xi) + \frac{1-\mu}{\xi} Z_4'(\xi)\right\}\right] \ldots \quad (40)$$

$$m_t = \frac{D}{L^2}\left[A\left\{\mu Z_4(\xi) + \frac{1-\mu}{\xi} Z_3'(\xi)\right\} + B\left\{-\mu Z_3(\xi) + \frac{1-\mu}{\xi} Z_4'(\xi)\right\}\right] \quad \ldots \quad (41)$$

and

$$v = \frac{D}{L^3}\left[-A Z_4'(\xi) + B Z_3'(\xi)\right] \quad \ldots\ldots\ldots \quad (42)$$

in which

$$A = \left[\frac{\Delta P}{2\pi k L^2}\right]$$

$$\left[\frac{\frac{1}{\beta}\left\{Z_3 - \frac{\beta^2}{6} Z_4 + \left(\frac{1-\mu}{\beta} + \frac{\beta^3}{12}\right) Z_4'\right\} - \left(1 - \frac{2}{3}\frac{\alpha}{\beta}\right)\left\{Z_3' - \frac{\beta}{2} Z_4 + \frac{\beta^2}{6} Z_4'\right\}}{\frac{\beta}{2}\left(Z_3^2 + Z_4^2\right) - \frac{\beta^2}{3}\left(Z_3 Z_3' + Z_4 Z_4'\right) + \left(\frac{3-\mu}{2} + \frac{\beta^4}{72}\right)\left(Z_3 Z_4' - Z_3' Z_4\right) + \left(\frac{1-\mu}{\beta} + \frac{\beta^3}{12}\right)\left(Z_3'^2 + Z_4'^2\right)}\right] \quad \ldots \quad (43)$$

and

$$B = \left[\frac{\Delta P}{2\pi k L^2}\right]$$

$$\left[\frac{-\left(1 - \frac{2}{3}\frac{\alpha}{\beta}\right)\left\{\frac{\beta}{2} Z_3 - \frac{\beta^2}{6} Z_3' + Z_4'\right\} + \frac{1}{\beta}\left\{\frac{\beta^2}{6} Z_3 - \left(\frac{1-\mu}{\beta} + \frac{\beta^3}{12}\right) Z_3' + Z_4\right\}}{\frac{\beta}{2}\left(Z_3^2 + Z_4^2\right) - \frac{\beta^2}{3}\left(Z_3 Z_3' + Z_4 Z_4'\right) + \left(\frac{3-\mu}{2} + \frac{\beta^4}{72}\right)\left(Z_3 Z_4' - Z_3' Z_4\right) + \left(\frac{1-\mu}{\beta} + \frac{\beta^3}{12}\right)\left(Z_3'^2 + Z_4'^2\right)}\right] \quad \ldots \quad (44)$$

in which y_b is given by Eq. 29, Θ_b by Eq. 30, ξ by Eq. 8a, α by Eq. 8b, β by Eq. 8c; and a is the radius of the loading plate, b represents the radius of the cracked zone, L denotes the radius of the relative stiffness, and Z represents the Z-functions. It is to be understood that β is the argument of the Z-functions in Eqs. 43 and 44.

DETAILED PROCEDURE

The deflection and moment distributions of the slab after bottom cracking, as well as the load causing top cracking, can be determined by means of superposition, as shown in the following procedure:

1. Compute the moment distributions before bottom surface cracking occurs and for different sizes of cracked zones $(b = b_o, b_1, b_2 \ldots)$ after cracking.
2. Determine the bottom cracking load, P_c, by the elastic moment distribution so that the tangential moment within a circular area exceeds the cracking moment. This circular area may be chosen equal to the loading area and the radius denoted by $b_o = a$ (see Fig. 1).
3. Compute the first load increment ΔP_1 such that the tangential moment becomes equal to the cracking moment for the circle of which the radius is equal to b_1 from

$$\Delta P_1 = \frac{m_c - m_{to}\,(\text{at } b_1)}{\bar{m}_{t1}\,(\text{at } b_1)} \quad \ldots\ldots\ldots\ldots \quad (45)$$

in which m_c represents the cracking moment of the slab, m_{to} is the elastic tangential moment due to the load P_c, and $\bar{m}_{t1}$ denotes the plastic tangential moment caused by the unit load that is obtained in the preliminary calculations for the cracked slab whose radius of cracked zone is equal to b_o.

4. Compute the moment distribution from the following:

$$m_r = m_{ro} + \Delta P_1 \bar{m}_{r1} \quad \ldots\ldots\ldots\ldots \quad (46a)$$

and

$$m_t = m_{to} + \Delta P_1 \bar{m}_{t1} \quad \ldots\ldots\ldots\ldots \quad (46b)$$

in which m_{ro} is the elastic radial moment due to the load P_c, and $\bar{m}_{ri}$ represents the plastic radial moment caused by unit load.

5. Compute the second load increment ΔP_2 such that the tangential moment becomes equal to the cracking moment for the circle whose radius is equal to b_2 from

$$\Delta P_2 = \frac{m_c - \left(m_{to} + \Delta P_1 \bar{m}_{t1}\right)(\text{at } b_2)}{\bar{m}_{t2}\,(\text{at } b_2)} \quad \ldots\ldots \quad (47)$$

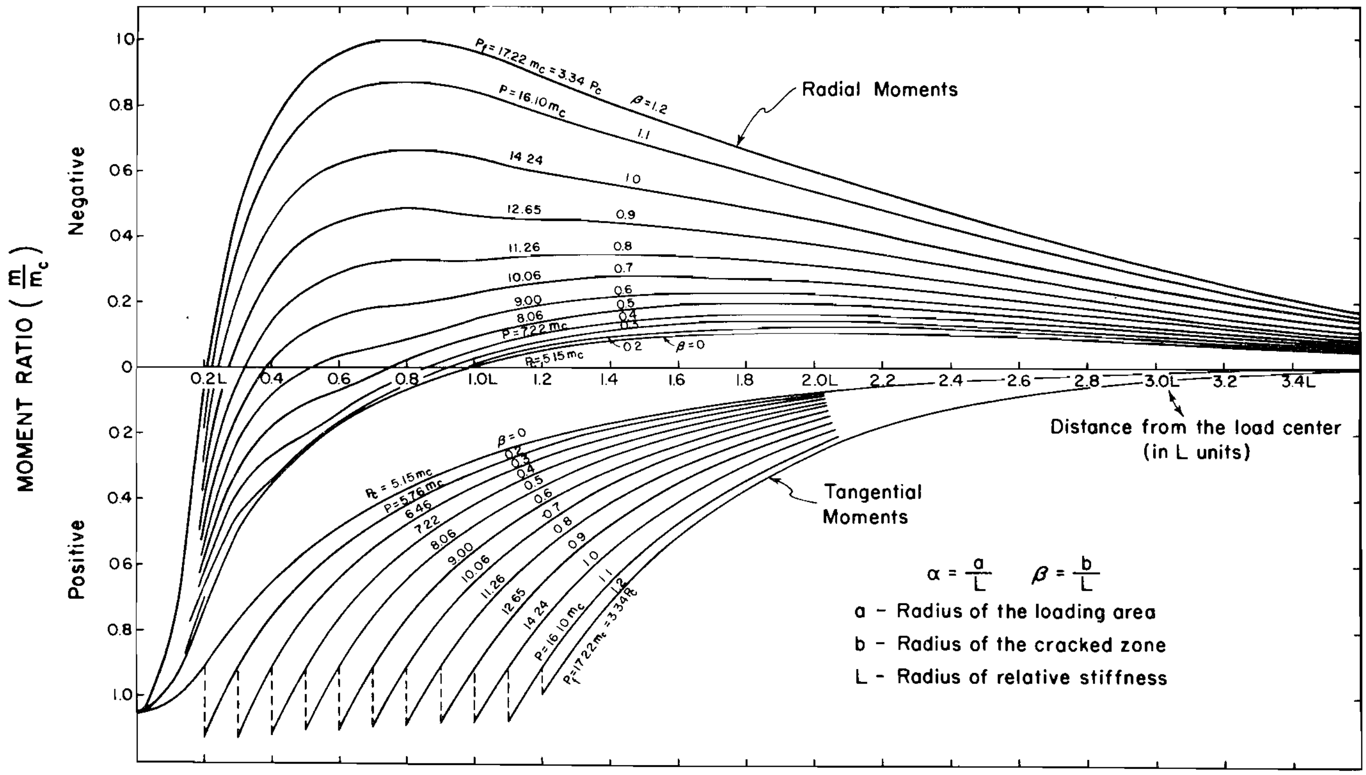

FIG. 6.—RADIAL AND TANGENTIAL MOMENT CURVES ($\alpha = 0.2$)

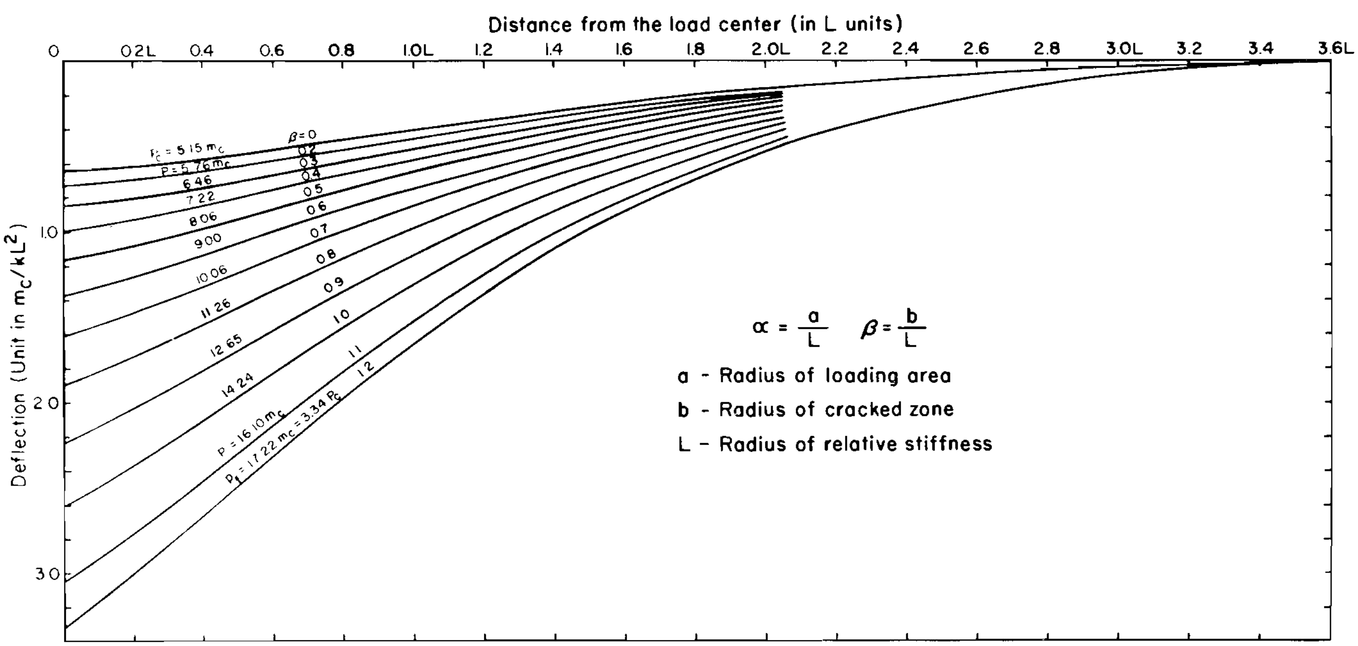

FIG. 7.—DEFLECTION CURVES ($\alpha = 0.2$)

6. Compute the moment distribution from the following:

$$m_r = \left(m_{ro} + \Delta P_1 \bar{m}_{r1}\right) + \Delta P_2 \bar{m}_{r2} \quad \ldots\ldots\ldots \quad (48a)$$

and

$$m_t = \left(m_{to} + \Delta P_1 \bar{m}_{t1}\right) + \Delta P_2 \bar{m}_{t2} \quad \ldots\ldots\ldots \quad (48b)$$

7. Continue the above procedure until the negative radial moment becomes equal to the cracking moment (see Fig. 1).

In step 2, P_c may be determined so that the average tangential moment within a circular area becomes equal to the cracking moment. A similar procedure may then be applied to determine each load increment ΔP (steps 3, 4, and so forth). This alternate method will yield a closer approximation of the load causing the top surface cracking. The alternate method was used in the following sample calculation.

SAMPLE CALCULATION

A sample calculation is given for the particular circular loading area having a radius equal to 0.2L.

The radial and tangential moment distributions are shown in Fig. 6 for each loading step, including that which caused top surface cracking. The deflection distributions are shown in Fig. 7. In Figs. 6 and 7, the applied load, P, the load causing the initial bottom surface cracking, P_c, and the load causing top surface cracking, P_f, are all expressed in terms of the cracking moment m_c. The distance from the load center is expressed in units of L in which

$$L = 4\sqrt{\frac{E\,h^3}{12\left(1-\mu^2\right)k}} \quad \ldots\ldots\ldots\ldots \quad (49)$$

In this calculation, the radius of the bottom cracked zone (b) was chosen in steps of 0.1L from 0.2L to 1.2L. From the results, the top cracking load, P_f, equals 3.34 P_c. The radius of the top crack, d, equals 0.8L, as determined graphically from Fig. 6.

CONCLUSIONS

This solution for the response of a prestressed concrete pavement to a load applied at an interior position provides numerical values of deflections and radial and tangential moments throughout the pavement for each load increment. Within the limitations of the assumptions defined in the text, the results indicate that prestressed concrete pavements are able to support loads greatly in excess of those that cause radial bottom surface cracks. This method is not proposed as a design procedure, but rather as a contribution toward a better understanding of the performance of prestressed pavements.

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APPENDIX I.—SOLUTION OF THE ELASTIC SLAB DUE TO CIRCULAR PLATE LOADING

Inside the Loading Area.—The fundamental differential equation is

$$D\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}\right)\left(\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx}\right) + ky = p \quad \ldots \quad (50)$$

in which

$$D = \frac{Eh^3}{\left[12\left(1-\mu^2\right)\right]} \quad \ldots\ldots\ldots\ldots\ldots \quad (51)$$

and is the flexural rigidity of the slab, x is the distance from the load center, y represents the deflection of the slab, k is the modulus of subgrade reaction, and p denotes the applied load per unit area. Substitution of $L^4 = \frac{D}{k}$ and $\xi = \frac{x}{L}$ into Eq. 50 yields

$$\left(\frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi}\right)\left(\frac{d^2y}{d\xi^2} + \frac{1}{\xi}\frac{dy}{d\xi}\right) + y = \frac{p}{k} \quad \ldots \quad (52)$$

The solution of Eq. 52 is given in terms of the Z-functions as

$$y = A_1 Z_1(\xi) + A_2 Z_2(\xi) + A_3 Z_3(\xi) + A_4 Z_4(\xi) + \frac{p}{k} \quad \ldots \quad (53)$$

The slope and the shear at the center of the loading area must be zero; therefore

$$A_3 = A_4 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (54)$$

and

$$y = A_1 Z_1(\xi) + A_2 Z_2(\xi) + \frac{p}{k} \quad \ldots\ldots\ldots \quad (55)$$

Accordingly,

$$L\frac{dy}{dx} = A_1 Z_1'(\xi) + A_2 Z_2'(\xi) \quad \ldots\ldots\ldots \quad (56)$$

$$m_r = \frac{D}{L^2}\left[A_1\left\{-Z_2(\xi) + \frac{1-\mu}{\xi} Z_1'(\xi)\right\} + A_2\left\{Z_1(\xi) + \frac{1-\mu}{\xi} Z_2'(\xi)\right\}\right]. \quad (57)$$

and

$$v = \frac{D}{L^3}\left[-A_1 Z_2'(\xi) + A_2 Z_1'(\xi)\right] \quad \ldots\ldots\ldots \quad (58)$$

Outside the Loading Area.—The fundamental differential equation is obtained by having p = 0 in Eq. 50; thus

$$D\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}\right)\left(\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx}\right) + ky = 0 \quad \ldots \quad (59)$$

The solution of this differential equation is

$$y = B_1 Z_1(\xi) + B_2 Z_2(\xi) + B_3 Z_3(\xi) + B_4 Z_4(\xi) \quad \ldots\ldots \quad (60)$$

Since the slab is of infinite extent, the deflection and slope approach zero as x approaches infinity; therefore,

$$B_1 = B_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (61)$$

$$y = B_3 Z_3(\xi) + B_4 Z_4(\xi) \quad \ldots\ldots\ldots\ldots \quad (62)$$

$$L\frac{dy}{dx} = B_3 Z_3'(\xi) + B_4 Z_4'(\xi) \quad \ldots\ldots\ldots \quad (63)$$

$$m_r = \frac{D}{L^2}\left[B_3\left\{-Z_4(\xi) + \frac{1-\mu}{\xi} Z_3'(\xi)\right\} + B_4\left\{Z_3(\xi) + \frac{1-\mu}{\xi} Z_4'(\xi)\right\}\right] .. \quad (64)$$

and

$$v = \frac{D}{L^3}\left[-B_3 Z_4'(\xi) + B_4 Z_3'(\xi)\right] \quad \ldots\ldots\ldots \quad (65)$$

Boundary Conditions.—At the boundary inside and outside the loading area, the deflections, slopes, moments, and shears must be the same; hence,

$$A_1 Z_1(\alpha) + A_2 Z_2(\alpha) + \frac{p}{k} = B_3 Z_3(\alpha) + B_4 Z_4(\alpha) \quad \ldots \quad (66)$$

$$A_1 Z_1'(\alpha) + A_2 Z_2'(\alpha) = B_3 Z_3'(\alpha) + B_4 Z_4'(\alpha) \quad \ldots \quad (67)$$

$$A_1\left\{-Z_2(\alpha) + \frac{1-\mu}{\alpha} Z_1'(\alpha)\right\} + A_2\left\{Z_1(\alpha) + \frac{1-\mu}{\alpha} Z_2'(\alpha)\right\}$$

$$= B_3\left\{-Z_4(\alpha) + \frac{1-\mu}{\alpha} Z_3'(\alpha)\right\} + B_4\left\{Z_3(\alpha) + \frac{1-\mu}{\alpha} Z_4'(\alpha)\right\} \quad \ldots \quad (68)$$

and

$$-A_1 Z_2'(\alpha) + A_2 Z_1'(\alpha) = -B_3 Z_4'(\alpha) + B_4 Z_3'(\alpha) \quad \ldots\ldots \quad (69)$$

Eq. 68 can be replaced by

$$-A_1 Z_2(\alpha) + A_2 Z_1(\alpha) = -B_3 Z_4(\alpha) + B_4 Z_3(\alpha) \quad \ldots\ldots \quad (70)$$

By solving Eqs. 66, 67, 69, and 70, simultaneously, the constants A_1, A_2, B_3, and B_4 are obtained. In order to solve the simultaneous equation the following relations between the Z-functions are used:

$$Z_1(\alpha) Z_3'(\alpha) - Z_2(\alpha) Z_4'(\alpha) - Z_1'(\alpha) Z_3(\alpha) + Z_2'(\alpha) Z_4(\alpha) = 0 \ldots (71)$$

and

$$Z_1(\alpha) Z_4'(\alpha) + Z_2(\alpha) Z_3'(\alpha) - Z_1'(\alpha) Z_4(\alpha) - Z_2'(\alpha) Z_3(\alpha) = \frac{2}{\pi\alpha} \; .. \; (72)$$

Thus,

$$A_1 = -\frac{\pi\alpha}{2} Z_4'(\alpha)\frac{p}{k} = -\frac{Z_4'(\alpha)}{2\alpha}\frac{P}{kL^2} \quad \ldots \quad (73)$$

$$A_2 = -\frac{Z_3'(\alpha)}{2\alpha}\frac{P}{kL^2} \quad \ldots\ldots\ldots\ldots \quad (74)$$

$$B_3 = -\frac{Z_2'(\alpha)}{2\alpha}\frac{P}{kL^2} \quad \ldots\ldots\ldots\ldots \quad (75)$$

and

$$B_4 = -\frac{Z_1'(\alpha)}{2\alpha}\frac{P}{kL^2} \quad \ldots\ldots\ldots\ldots \quad (76)$$

Final Expressions.—Inside the loading area ($x \leq a$):

$$y = \frac{P}{2kL^2\alpha}\left[\frac{2}{\pi\alpha} - Z_4'(\alpha) Z_1(\xi) - Z_3'(\alpha) Z_2(\xi)\right] \quad \ldots \quad (77)$$

$$\frac{dy}{dx} = \frac{P}{2\,k\,L^3 \alpha}\left[-Z_4'(\alpha)Z_1'(\xi) - Z_3'(\alpha)Z_2'(\xi)\right] \quad \ldots\ldots \quad (78)$$

$$m_r = \frac{P}{2\alpha}\left[-Z_4'(\alpha)\left\{-Z_2(\xi) + \frac{1-\mu}{\xi}Z_1'(\xi)\right\} - Z_3'(\alpha)\left\{Z_1(\xi) + \frac{1-\mu}{\xi}Z_2'(\xi)\right\}\right] \quad .. \quad (79)$$

$$m_t = \frac{P}{2\alpha}\left[Z_4'(\alpha)\left\{\mu\, Z_2(\xi) + \frac{1-\mu}{\xi}Z_1'(\xi)\right\} + Z_3'(\alpha)\left\{-\mu\, Z_1(\xi) + \frac{1-\mu}{\xi}Z_2'(\xi)\right\}\right]. \quad (80)$$

and

$$v = \frac{P}{2\,L\alpha}\left[Z_4'(\alpha)\,Z_2'(\xi) - Z_3'(\alpha)\,Z_1'(\xi)\right] \quad \ldots\ldots \quad (81)$$

Outside the loading area ($x \geq a$):

$$y = \frac{P}{2\,k\,L^2 \alpha}\left[-Z_2'(\alpha)\,Z_3(\xi) - Z_1'(\alpha)\,Z_4(\xi)\right] \quad \ldots\ldots\ldots \quad (82)$$

$$\frac{dx}{dx} = \frac{P}{2\,k\,L^3 \alpha}\left[-Z_2'(\alpha)\,Z_3'(\xi) - Z_1'(\alpha)Z_4'(\xi)\right] \quad \ldots\ldots \quad (83)$$

$$m_r = \frac{P}{2\alpha}\left[-Z_2'(\alpha)\left\{-Z_4(\xi) + \frac{1-\mu}{\xi}Z_3'(\xi)\right\} - Z_1'(\alpha)\left\{Z_3(\xi) + \frac{1-\mu}{\xi}Z_4'(\xi)\right\}\right] \quad .. \quad (84)$$

$$m_t = \frac{P}{2\alpha}\left[Z_2'(\alpha)\left\{\mu\, Z_4(\xi) + \frac{1-\mu}{\xi}Z_3'(\xi)\right\} + Z_1'(\alpha)\left\{-\mu\, Z_3(\xi) + \frac{1-\mu}{\xi}Z_4'(\xi)\right\}\right] \quad .. \quad (85)$$

and

$$v = \frac{P}{2\,L\alpha}\left[Z_2'(\alpha)Z_4'(\xi) - Z_1'(\alpha)Z_3'(\xi)\right] \quad \ldots\ldots \quad (86)$$

APPENDIX II.—NOTATION

The following symbols have been adopted for use in this paper:

A = simplifying function (see Eq. 43);

A_1, A_2, A_3, A_4 = constants of integration;

a = radius of circular area over which load is uniformly distributed;

B = simplifying function (see Eq. 44);

B_1, B_2, B_3, B_4 = constants of integration;

b, b_o, b_1, b_2 = radii of circular areas defining the stepwise limits of bottom surface radial cracks;

C_1, C_2, C_3, C_4 = constants of integration;

D = flexural rigidity of the slab or $k\,L^4$;

d = radius of the top surface crack;

E = elastic modulus of concrete;

$H_o^{(1)}$ = Hankel function;

h = thickness of the slab;

J_o = Bessel function of the first kind;

k = modulus of subgrade reaction;

L = radius of relative stiffness or $\sqrt[4]{\dfrac{E\,h^3}{12\left(1-\mu^2\right)k}}$

M_o = radial bending moment due to applied load;

M_1 = radial bending moment due to subgrade reaction;

M_r = total radial bending moment;

m_c = cracking moment of the slab;

m_r = radial bending moment per unit length of arc;

m_{ro} = elastic radial moment due to load P_c;

$\bar{m}_{r1}, \bar{m}_{r2}$ = plastic radial moment at intermediate steps;

m_t = tangential bending moment per unit length of arc;

m_{to} = elastic tangential moment due to load P_c;

$\bar{m}_{t1}, \bar{m}_{t2}$ = plastic tangential moment at intermediate steps;

P_c = load causing bottom surface cracking;

P_f = load causing top surface cracking;

$\Delta P, \Delta P_1, \Delta P_2$ = increments of the total load;

p = load per unit area;

ΔP = increment of load per unit area;

r = variable distance from the apex, for integration;

v = shearing force per unit length of arc;

x = variable distance from center of loaded area;

y = deflection of the slab at any point;

y_b = deflection of the slab at x = b;

Z_1, Z_2, Z_3, Z_4 = Z functions (real and imaginary parts of Bessel and Hankel functions);

Z_1', Z_2', Z_3', Z_4' = first derivative of Z-functions;

α = $\frac{a}{L}$;

β = $\frac{b}{L}$;

Θ_b = slope of the deflection line at x = b;

μ = Poisson's ratio of concrete; and

ξ = $\frac{x}{L}$.